

Solved Exercises of G.J. Mellenbergh's (2011)
A conceptual introduction to psychometrics:
Development, analysis, and application of psychological and educational tests

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Chapter 2

$$2.4 \quad O = \frac{10}{103} + \frac{20}{103} + \frac{18}{103} + \frac{8}{103} + \frac{7}{103} = \frac{63}{103} = .612.$$

$$E = \frac{18}{103} \times \frac{15}{103} + \frac{31}{103} \times \frac{31}{103} + \frac{29}{103} \times \frac{30}{103} + \frac{15}{103} \times \frac{17}{103} + \frac{10}{103} \times \frac{10}{103} = \frac{18 \times 15 + 31 \times 31 + 29 \times 30 + 15 \times 17 + 10 \times 10}{103 \times 103} = .232.$$

$$\text{Kappa} = \frac{O - E}{1 - E} = \frac{.612 - .232}{1 - .232} = .49.$$

Chapter 3

3.4 Nurse A's ratings, squared ratings, and sum of her squared ratings are given in Table 3.3.

The sum of her squared ratings is:

$$\sum_{j=1}^{10} R_{1j}^2 = 418.78.$$

The ratings (R_1), reratings (R_2), squared reratings (R_2^2), and product of the ratings and reratings (R_1R_2) are:

Patient	R_1	R_2	R_2^2	R_1R_2
1	2.7	4.6	21.16	12.42
2	7.5	6.1	37.21	45.75
3	9.3	7.5	56.25	69.75
4	4.4	7.3	53.29	32.12
5	5.7	6.2	38.44	35.34
6	8.1	7.1	50.41	57.51
7	3.2	6.2	38.44	19.84
8	7.6	5.7	32.49	43.32
9	7.5	6.0	36.00	45.00
10	5.2	7.3	53.29	37.96

$$\sum_{j=1}^{10} R_{2j}^2 = 416.98$$

$$\sum_{j=1}^{10} R_{1j}R_{2j} = 399.01$$

$$ID(C) = \frac{2 \sum_{j=1}^{10} R_{1j} R_{2j}}{\sum_{j=1}^{10} R_{1j}^2 + \sum_{j=1}^{10} R_{2j}^2} = \frac{2 \times 399.01}{418.78 + 416.98} = .95.$$

Chapter 4

4.1 $US_3 = 5 + 5 = 10.$

4.2 $WS_4 = 1 \times 2 + 2 \times 2 + 3 \times 1 = 9;$

$WS_5 = 1 \times 3 + 2 \times 2 + 3 \times 2 = 13.$

4.3 The personal mean scores are:

$$PM_1 = \frac{6+5+6+7}{4} = \frac{24}{4} = 6.00,$$

$$PM_2 = \frac{2+3+3}{3} = \frac{8}{3} = 2.67,$$

$$PM_3 = \frac{3+4+4+3}{4} = \frac{14}{4} = 3.50,$$

$$PM_4 = \frac{5+5+7+6}{4} = \frac{23}{4} = 5.75,$$

$$PM_5 = \frac{6+5+7}{3} = \frac{18}{3} = 6.00,$$

$$PM_6 = \frac{2+1+3+2}{4} = \frac{8}{4} = 2.00;$$

the item mean scores are:

$$IM_1 = \frac{6+2+3+5+2}{5} = \frac{18}{5} = 3.60,$$

$$IM_2 = \frac{5+4+5+6+1}{5} = \frac{21}{5} = 4.20,$$

$$IM_3 = \frac{6+3+4+7+5+3}{6} = \frac{28}{6} = 4.67,$$

$$IM_4 = \frac{7+3+3+6+7+2}{6} = \frac{28}{6} = 4.67;$$

the overall mean is the mean of all item scores

$$OM = \frac{6+5+\dots+3+2}{22} = \frac{95}{22} = 4.32.$$

$$TW_{22} = PM_2 + IM_2 - OM = 2.67 + 4.20 - 4.32 = 2.55;$$

$$TW_{51} = PM_5 + IM_1 - OM = 6.00 + 3.60 - 4.32 = 5.28.$$

The estimated E-values of the observed item scores are:

$$\hat{E}_{11} = 6 - TW_{11} = 6 - (PM_1 + IM_1 - OM) = 6.00 - (6.00 + 3.60 - 4.32) = .72,$$

$$\hat{E}_{12} = 5 - TW_{12} = 5 - (PM_1 + IM_2 - OM) = 5.00 - (6.00 + 4.20 - 4.32) = -.88,$$

$$\hat{E}_{13} = 6 - TW_{13} = 6 - (PM_1 + IM_3 - OM) = 6.00 - (6.00 + 4.67 - 4.32) = -.35,$$

$$\hat{E}_{14} = 7 - TW_{14} = 7 - (PM_1 + IM_4 - OM) = 7.00 - (6.00 + 4.67 - 4.32) = .65,$$

$$\hat{E}_{21} = 2 - TW_{21} = 2 - (PM_2 + IM_1 - OM) = 2.00 - (2.67 + 3.60 - 4.32) = .05,$$

$$\hat{E}_{23} = 3 - TW_{23} = 3 - (PM_2 + IM_3 - OM) = 3.00 - (2.67 + 4.67 - 4.32) = -.02,$$

$$\hat{E}_{24} = 3 - TW_{24} = 3 - (PM_2 + IM_4 - OM) = 3.00 - (2.67 + 4.67 - 4.32) = -.02,$$

$$\hat{E}_{31} = 3 - TW_{31} = 3 - (PM_3 + IM_1 - OM) = 3.00 - (3.50 + 3.60 - 4.32) = .22,$$

$$\hat{E}_{32} = 4 - TW_{32} = 4 - (PM_3 + IM_2 - OM) = 4.00 - (3.50 + 4.20 - 4.32) = .62,$$

$$\hat{E}_{33} = 4 - TW_{33} = 4 - (PM_3 + IM_3 - OM) = 4.00 - (3.50 + 4.67 - 4.32) = .15,$$

$$\hat{E}_{34} = 3 - TW_{34} = 3 - (PM_3 + IM_4 - OM) = 3.00 - (3.50 + 4.67 - 4.32) = -.85,$$

$$\hat{E}_{41} = 5 - TW_{41} = 5 - (PM_4 + IM_1 - OM) = 5.00 - (5.75 + 3.60 - 4.32) = -.03,$$

$$\hat{E}_{42} = 5 - TW_{42} = 5 - (PM_4 + IM_2 - OM) = 5.00 - (5.75 + 4.20 - 4.32) = -.63,$$

$$\hat{E}_{43} = 7 - TW_{43} = 7 - (PM_4 + IM_3 - OM) = 7.00 - (5.75 + 4.67 - 4.32) = .90,$$

$$\hat{E}_{44} = 6 - TW_{44} = 6 - (PM_4 + IM_4 - OM) = 6.00 - (5.75 + 4.67 - 4.32) = -.10,$$

$$\hat{E}_{52} = 6 - TW_{52} = 6 - (PM_5 + IM_2 - OM) = 6.00 - (6.00 + 4.20 - 4.32) = .12,$$

$$\hat{E}_{53} = 5 - TW_{53} = 5 - (PM_5 + IM_3 - OM) = 5.00 - (6.00 + 4.67 - 4.32) = -1.35,$$

$$\hat{E}_{54} = 7 - TW_{54} = 7 - (PM_5 + IM_4 - OM) = 7.00 - (6.00 + 4.67 - 4.32) = .65,$$

$$\hat{E}_{61} = 2 - TW_{61} = 2 - (PM_6 + IM_1 - OM) = 2.00 - (2.00 + 3.60 - 4.32) = .72,$$

$$\hat{E}_{62} = 1 - TW_{62} = 1 - (PM_6 + IM_2 - OM) = 1.00 - (2.00 + 4.20 - 4.32) = -.88,$$

$$\hat{E}_{63} = 3 - TW_{63} = 3 - (PM_6 + IM_3 - OM) = 3.00 - (2.00 + 4.67 - 4.32) = .65,$$

$$\hat{E}_{64} = 2 - TW_{64} = 2 - (PM_6 + IM_4 - OM) = 2.00 - (2.00 + 4.67 - 4.32) = -.35.$$

The variances of the \hat{E} -values is

$$\frac{(.72)^2 + (-.88)^2 + \dots + (.65)^2 + (-.35)^2}{22 - 1} = .3969$$

A value of $E_{22} = .71$ was randomly selected from a normal distribution with mean 0 and variance .40; the imputed item score of the second test taker is

$$2.55 + .71 = 3.26, \text{ rounded to } 3.$$

Another value of $E_{51} = -.21$ was randomly selected from the normal distribution with mean 0 and variance .40; the imputed item score of the fifth test taker is

$$5.28 - .21 = 5.07, \text{ rounded to } 5.$$

4.4 The test taker's item scores are

Item no.	Score indicative item	Score contra-indicative item
1	3	
2		4
3	5	
4		5
5	4	
6	5	
7	5	
8		3
9	4	
10		5

The test taker's personal mean score on the indicative items is

$$\text{PMI} = \frac{3 + 5 + 4 + 5 + 5 + 4}{6} = 4.33, \text{ and his (her) personal mean score on the contra-indicative$$

items is

$$\text{PMC} = \frac{4 + 5 + 3 + 5}{4} = 4.25.$$

The values of PMI and PMC are approximately equal, which indicates that the test taker is not yea-saying or nay-saying to the items.

Chapter 5

5.1

$$\begin{aligned} \text{Alpha} \leq \text{Rel}(S) &\Rightarrow (1 - \text{Alpha}) \geq \{1 - \text{Rel}(S)\} \Rightarrow \text{Var}_p(S)(1 - \text{Alpha}) \geq \text{Var}_p(S)\{1 - \text{Rel}(S)\} \Rightarrow \\ &\sqrt{\text{Var}_p(S)(1 - \text{Alpha})} \geq \sqrt{\text{Var}_p(S)\{1 - \text{Rel}(S)\}}. \end{aligned}$$

5.2

$$\begin{aligned} \text{Rel}\{S(K)\} &= \frac{K \text{Rel}(S)}{1 + (K - 1)\text{Rel}(S)} \Rightarrow K \text{Rel}(S) = \text{Rel}\{S(K)\}\{1 + (K - 1)\text{Rel}(S)\} \Rightarrow \\ K \text{Rel}(S) &= \text{Rel}\{S(K)\} + K \text{Rel}\{S(K)\}\text{Rel}(S) - \text{Rel}\{S(K)\}\text{Rel}(S) \Rightarrow \\ K \text{Rel}(S) - K \text{Rel}\{S(K)\}\text{Rel}(S) &= \text{Rel}\{S(K)\} - \text{Rel}\{S(K)\}\text{Rel}(S) \Rightarrow \\ K \text{Rel}(S)[1 - \text{Rel}\{S(K)\}] &= \text{Rel}\{S(K)\}\{1 - \text{Rel}(S)\} \Rightarrow K = \frac{\{1 - \text{Rel}(S)\}\text{Rel}\{S(K)\}}{[1 - \text{Rel}\{S(K)\}]\text{Rel}(S)}. \end{aligned}$$

Chapter 6

$$\begin{aligned} 6.1(a) \quad \overline{US} &= \frac{1}{N_s} \sum_{j=1}^{N_s} US_j = \frac{1}{N_s} \sum_{j=1}^{N_s} (X_{j1} + X_{j2} + \dots + X_{jn}) = \frac{1}{N_s} \sum_{j=1}^{N_s} X_{j1} + \frac{1}{N_s} \sum_{j=1}^{N_s} X_{j2} + \dots + \frac{1}{N_s} \sum_{j=1}^{N_s} X_{jn} = \\ &\overline{X}_1 + \overline{X}_2 + \dots + \overline{X}_n = \sum_{k=1}^n \overline{X}_k. \end{aligned}$$

$$\begin{aligned} 6.1(b) \quad \overline{WS} &= \frac{1}{N_s} \sum_{j=1}^{N_s} WS_j = \frac{1}{N_s} \sum_{j=1}^{N_s} (w_1 X_{j1} + w_2 X_{j2} + \dots + w_n X_{jn}) = \\ &\frac{1}{N_s} \sum_{j=1}^{N_s} w_1 X_{j1} + \frac{1}{N_s} \sum_{j=1}^{N_s} w_2 X_{j2} + \dots + \frac{1}{N_s} \sum_{j=1}^{N_s} w_n X_{jn} = \end{aligned}$$

$$w_1 \frac{1}{N_s} \sum_{j=1}^{N_s} X_{j1} + w_2 \frac{1}{N_s} \sum_{j=1}^{N_s} X_{j2} + \dots + w_n \frac{1}{N_s} \sum_{j=1}^{N_s} X_{jn} = w_1 \bar{X}_1 + w_2 \bar{X}_2 + \dots + w_n \bar{X}_n = \sum_{k=1}^n w_k \bar{X}_k.$$

6.2 For dichotomously scored items $X_{jk}^2 = X_{jk}$ and $\bar{X}_k = \frac{\sum_{j=1}^{N_s} X_{jk}}{N_s} = p_k$ because X_{jk} is 0 or 1.

Therefore, for dichotomously scored items

$$\begin{aligned} S_k^2 &= \frac{\sum_{j=1}^{N_s} (X_{jk} - \bar{X}_k)^2}{N_s - 1} = \frac{\sum_{j=1}^{N_s} (X_{jk}^2 - 2\bar{X}_k X_{jk} + \bar{X}_k^2)}{N_s - 1} = \frac{\sum_{j=1}^{N_s} X_{jk}^2 - \sum_{j=1}^{N_s} 2\bar{X}_k X_{jk} + \sum_{j=1}^{N_s} \bar{X}_k^2}{N_s - 1} = \\ &= \frac{\sum_{j=1}^{N_s} X_{jk}^2 - 2\bar{X}_k \sum_{j=1}^{N_s} X_{jk} + \sum_{j=1}^{N_s} \bar{X}_k^2}{N_s - 1} = \frac{\sum_{j=1}^{N_s} X_{jk} - 2\bar{X}_k \sum_{j=1}^{N_s} X_{jk} + \sum_{j=1}^{N_s} \bar{X}_k^2}{N_s - 1} = \\ &= \frac{N_s \bar{X}_k - 2\bar{X}_k N_s \bar{X}_k + N_s \bar{X}_k^2}{N_s - 1} = \frac{N_s \bar{X}_k - 2N_s \bar{X}_k^2 + N_s \bar{X}_k^2}{N_s - 1} = \frac{N_s \bar{X}_k - N_s \bar{X}_k^2}{N_s - 1} = \\ &= \frac{N_s \bar{X}_k (1 - \bar{X}_k)}{N_s - 1} = \frac{N_s}{N_s - 1} p_k (1 - p_k). \end{aligned}$$

6.3 $\hat{\text{Var}}_p(S^{(3)}) = \hat{\text{Var}}_p(S - 3) = \hat{\text{Var}}_p(S).$

6.4 Taking the derivative of Formula (6.10), and setting the result equal to 0 yields.

$$\frac{N_s}{N_s - 1} (1 - 2p_k) = 0 \Rightarrow 2p_k = 1 \Rightarrow p_k = .5.$$

6.5(a) $r_{x_1s} = .743, r_{x_2s} = .796, r_{x_3s} = .689;$

6.5(b) $r_{x_1s^{(1)}} = .384,$

Test taker no.	S ⁽²⁾	S ⁽³⁾
1	2	2
2	1	1
3	2	2
4	2	1
5	1	1
6	2	2
7	0	0
8	1	2
9	1	0
10	0	0

$r_{x_2s^{(2)}} = .491, r_{x_3s^{(3)}} = .342.$

6.5(c) $s_1 = \frac{10}{9} \times .5(1-.5) = .278,$

$s_2 = \frac{10}{9} \times .6(1-.6) = .267,$

$s_3 = \frac{10}{9} \times .7(1-.7) = .233;$

$\hat{IRI}(X_1) = .743 \times .278 = .21,$

$\hat{IRI}(X_2) = .796 \times .267 = .21,$

$\hat{IRI}(X_3) = .689 \times .233 = .16.$

6.7 Item 1 distractor a-rest correlation: .254,

Item 1 distractor c-rest correlation: .800,

Item 2 distractor rest correlations:

Test taker no.	Correct/ Distractor b	Correct/ Distractor c	b	S ⁽²⁾	c
1	1	1	2	2	2
2	1	1	1	1	1
3	1	1	2	2	2
4	0		2		

5	1	1	1	1
6	1	1	2	2
7		0		0
8	1	1	1	1
9	0		1	
10		0		0

Item 2 distractor b-rest correlation: 0,

Item 2 distractor c-rest correlation: .832;

Item 3 distractor rest correlations:

Test taker no.	Correct/ Distractor a	Correct/ Distractor b	S ⁽³⁾	
			a	b
1	1	1	2	2
2	1	1	1	1
3	1	1	2	2
4	1	1	1	1
5	1	1	1	1
6	1	1	2	2
7	0		0	
8		0		2
9	1	1	0	0
10	0		0	

Item 3 distractor a-rest correlation: .655,

Item 3 distractor b-rest correlation: .339.

$$6.8 \quad d_{12} + d_{34} = \sqrt{(3.83 - 3.56)^2 + (.52 - .53)^2} + \sqrt{(3.53 - 3.30)^2 + (.64 - .69)^2} = .51,$$

$$d_{13} + d_{24} = \sqrt{(3.83 - 3.53)^2 + (.52 - .64)^2} + \sqrt{(3.56 - 3.30)^2 + (.53 - .69)^2} = .63,$$

$$d_{14} + d_{23} = \sqrt{(3.83 - 3.30)^2 + (.52 - .69)^2} + \sqrt{(3.56 - 3.53)^2 + (.53 - .64)^2} = .67.$$

Optimal splitting: (1, 2) and (3, 4).

Subtests

(a) Subtests (1) 1 and 3 and (2) 2 and 4:

Item rest correlations 1 and 3: .43 + .47 = .90,

Item rest correlations 2 and 4: $.44 + .50 = .94$,

Difference $.94 - .90 = .04$.

(b) Subtests (1) 1 and 4 and (2) 2 and 3:

Item rest correlations 1 and 4: $.43 + .50 = .93$,

Item rest correlations 2 and 3: $.44 + .47 = .91$,

Difference $.93 - .91 = .02$.

Optimal subtests: Subtest 1: items 1 and 4,

Subtest 2: items 2 and 3.

6.9 For dichotomously scored items $\bar{X}_k = p_k$ and $\bar{X}_l = p_l$ (see Formula 6.3). Therefore, the numerator of Formula 6.18 is:

$$\begin{aligned} \sum_{j=1}^{N_s} (X_{jk} - \bar{X}_k)(X_{jl} - \bar{X}_l) &= \sum_{j=1}^{N_s} (X_{jk} - p_k)(X_{jl} - p_l) = \sum_{j=1}^{N_s} (X_{jk}X_{jl} - X_{jk}p_l - p_kX_{jl} + p_kp_l) = \\ &= \sum_{j=1}^{N_s} X_{jk}X_{jl} - p_l \sum_{j=1}^{N_s} X_{jk} - p_k \sum_{j=1}^{N_s} X_{jl} + \sum_{j=1}^{N_s} p_kp_l = \\ &= \left(\sum_{j=1}^{N_s} X_{jk} = N_s \bar{X}_k = N_s p_k \text{ and } \sum_{j=1}^{N_s} X_{jl} = N_s \bar{X}_l = N_s p_l \right) \\ &= \sum_{j=1}^{N_s} X_{jk}X_{jl} - N_s p_k p_l - N_s p_k p_l + N_s p_k p_l = \sum_{j=1}^{N_s} X_{jk}X_{jl} - N_s p_k p_l = \end{aligned}$$

(the product $X_{jk}X_{jl} = 1$ if both $X_{jk} = 1$ and $X_{jl} = 1$, otherwise this product is 0. Therefore,

$$\begin{aligned} \sum_{j=1}^{N_s} X_{jk}X_{jl} \text{ is the frequency of items that are both correctly (yes) answered, which is equal to} \\ N_s p_{kl} - N_s p_k p_l - N_s p_k p_l + N_s p_k p_l = N_s (p_{kl} - p_k p_l). \end{aligned} \quad (1)$$

The first term under the square root of the denominator of Formula 6.18 is:

$$\begin{aligned} \sum_{j=1}^{N_s} (X_{jk} - p_k)^2 &= \sum_{j=1}^{N_s} (X_{jk}^2 - 2p_k X_{jk} + p_k^2) = \sum_{j=1}^{N_s} X_{jk}^2 - 2p_k \sum_{j=1}^{N_s} X_{jk} + \sum_{j=1}^{N_s} p_k^2 = \\ &= \sum_{j=1}^{N_s} X_{jk}^2 - 2N_s p_k p_k + N_s p_k^2 = \sum_{j=1}^{N_s} X_{jk}^2 - 2N_s p_k^2 + N_s p_k^2 = \sum_{j=1}^{N_s} X_{jk}^2 - N_s p_k^2 = \end{aligned}$$

(for dichotomously (0/1) scored items $X_{jk}^2 = X_{jk}$ because X_{jk} is 0 or 1)

$$\sum_{j=1}^{N_s} X_{jk} - N_s p_k^2 = N_s \bar{X}_k - N_s p_k^2 = N_s p_k - N_s p_k^2 = N_s p_k (1 - p_k). \quad (2)$$

In the same way it is shown that the second term under the square root of Formula 6.18 is:

$$\sum_{j=1}^{N_s} (X_{j1} - p_1)^2 = N_s p_1 (1 - p_1). \quad (3)$$

Substituting (1), (2), and (3) into Formula 6.18 yields Formula 6.19:

$$r_{X_k X_1} = \frac{N_s (p_{kl} - p_p p_l)}{\sqrt{N_s p_k (1 - p_k) N_s p_l (1 - p_l)}} = \frac{N_s (p_{kl} - p_p p_l)}{\sqrt{N_s^2 p_k (1 - p_k) p_l (1 - p_l)}} = \frac{p_{kl} - p_p p_l}{\sqrt{p_k (1 - p_k) p_l (1 - p_l)}} = \hat{\varphi}_{kl}.$$

6.10

		Item 3			
		0	1		
Item 1	1	0	2	2	$p_{13} = 2/10 = .2$
	0	2	6		
		2	8	10	
		$p_3 = 8/10 = .8$			

$$\hat{\varphi}_{13} = \frac{.2 - .2 \times .8}{\sqrt{.2 \times (1 - .2) \times .8 \times (1 - .8)}} = .25$$

		Item 4			
		0	1		
Item 1	1	0	2	2	$p_{14} = 2/10 = .2$
	0	2	6		
		2	8	10	
		$p_4 = 8/10 = .8$			

$$\hat{\varphi}_{14} = \frac{.2 - .2 \times .8}{\sqrt{.2 \times (1 - .2) \times .8 \times (1 - .8)}} = .25$$

		Item 4		
		0	1	
Item 2	1	0	2 $p_{24} = 2/10 = .2$	2 $p_2 = 2/10 = .2$
	0	2	6	8
		2	8	10
		$p_4 = 8/10 = .8$		

$$\hat{\Phi}_{24} = \frac{.2 - .2 \times .8}{\sqrt{.2 \times (1 - .2) \times .8 \times (1 - .8)}} = .25$$

		Item 4		
		0	1	
Item 3	1	0	8 $p_{34} = 8/10 = .8$	8 $p_3 = 8/10 = .8$
	0	2	0	2
		2	8	10
		$p_4 = 8/10 = .8$		

$$\hat{\Phi}_{34} = \frac{.8 - .8 \times .8}{\sqrt{.8 \times (1 - .8) \times .8 \times (1 - .8)}} = 1$$

6.11

		U ₁	
		(i, c)	(p, c) (c, c)
		U ₁₁	
U _k	(i, p)	U _{k1}	0
		U ₁₂	U _{k2}
		(i, i)	(p, i) (c, i)

c: correct answer;
p: partly correct answer;
i: incorrect answer.

6.12 The test taker's tied-percentile ranks score at the item 'Squabbling' is

$$\frac{931+1258}{2} \times \frac{100}{1397} = 78.35, \text{ and}$$

his (her) tied-percentile ranks score at the item 'Unpleasant' is

$$\frac{1204+1350}{2} \times \frac{100}{1397} = 91.41.$$

The test taker's tied percentile ranks sum score is

$$78.35 + 91.41 = 169.8.$$

Chapter 7

7.1

Item	Response pattern			Probability
	1	2	3	
	c	c	c	$.5 \times .4 \times .3 = .06$
	c	c	i	$.5 \times .4 \times (1 - .3) = .14$
	c	i	c	$.5 (1 - .4) \times .3 = .09$
	c	i	i	$.5 (1 - .4) \times (1 - .3) = .21$
	i	c	c	$(1 - .5) \times .4 \times .3 = .06$
	i	c	i	$(1 - .5) \times .4 \times (1 - .3) = .14$
	i	i	c	$(1 - .5) \times (1 - .4) \times .3 = .09$
	i	i	i	$(1 - .5) \times (1 - .4) \times (1 - .3) = .21$

7.2 Table (a): Expected frequencies are equal to the observed frequencies, for example, the expected frequency of the (c, i) pattern is

$$\frac{(96+1824) \times (96+4)}{2000} = 96.$$

Therefore, $X^2 = 0$ (df = 1), and the null hypothesis that the responses to the two items are independently distributed is not rejected.

Table (b): Expected frequencies are equal to the observed frequencies, for example, the expected frequency of the (c, i) pattern is

$$\frac{(49 + 1) \times (49 + 931)}{1000} = 49.$$

Therefore, $X^2 = 0$ (df = 1), and the null hypothesis that the responses to the two items are independently distributed is not rejected.

Table (c): The expected frequencies of the responses patterns are:

$$(c, i): \frac{(145 + 1825) \times (145 + 935)}{3000} = 709.20,$$

$$(c, c): \frac{(1825 + 145) \times (1825 + 95)}{3000} = 1260.80,$$

$$(i, c): \frac{(95 + 1825) \times (95 + 935)}{3000} = 659.20,$$

$$(i, i): \frac{(935 + 95) \times (935 + 145)}{3000} = 370.80,$$

$$X^2 = \frac{(145 - 709.2)^2}{709.2} + \frac{(1825 - 1260.8)^2}{1260.8} + \frac{(95 - 659.2)^2}{659.2} + \frac{(935 - 370.8)^2}{370.8} = 2042.7.$$

At the 1% level and 1 degree of freedom a value larger than $X^2 = 6.64$ is significant. Therefore, the null hypothesis that the responses to the two items are independently distributed is rejected.

Chapter 8

8.1 Expected frequency (c, i) pattern:

$$110 \times \frac{2}{9} (1 - .799) = 4.5;$$

expected frequency (i, c) pattern:

$$110 \times \frac{2}{9}(1 - .799) = 4.5;$$

expected frequency (i, i) pattern:

$$110 \times \frac{4}{9}(1 - .799) = 8.9.$$

$$8.2 \quad \hat{Pr}_1(c, c, i, i, i) = .97 \times .97 \times (1 - .97) \times (1 - .97) \times (1 - .97) = .0000,$$

$$\hat{Pr}_2(c, c, i, i, i) = .04 \times .04 \times (1 - .04) \times (1 - .04) \times (1 - .04) = .0014,$$

$$\hat{Pr}_3(c, c, i, i, i) = .65 \times .65 \times (1 - .65) \times (1 - .65) \times (1 - .65) = .0181.$$

$$\hat{Pr}(LC = 1 | (c, c, i, i, i)) = \frac{.60 \times .0000}{.60 \times .0000 + .22 \times .0014 + .18 \times .0181} = .00,$$

$$\hat{Pr}(LC = 2 | (c, c, i, i, i)) = \frac{.22 \times .0014}{.60 \times .0000 + .22 \times .0014 + .18 \times .0181} = .09,$$

$$\hat{Pr}(LC = 3 | (c, c, i, i, i)) = \frac{.18 \times .0181}{.60 \times .0000 + .22 \times .0014 + .18 \times .0181} = .91.$$

The child is assigned to the third latent class.

- 8.3 The four items are ordered such that Item 1 is easier than Item 2, Item 2 is easier than Item 3, and Item 3 is easier than Item 4. The possible item response patterns are (c: correct or yes-answer; i: incorrect or no-answer):

Item no.	1	2	3	4	Admissable under Guttman model
	c	c	c	c	yes
	c	c	c	i	yes
	c	c	i	c	no
	c	c	i	i	yes
	c	i	c	c	no
	c	i	c	i	no
	c	i	i	c	no

c	i	i	i	yes
i	c	c	c	no
i	c	c	i	no
i	c	i	c	no
i	c	i	i	no
i	i	c	c	no
i	i	c	i	no
i	i	i	c	no
i	i	i	i	yes

8.4 The item response patterns (c: correct or yes-answer; i: incorrect or no-answer) and their probabilities are:

	Item no.			Probability
	1	2	3	
c	c	c	c	$(1 - m) \times (1 - m) \times (1 - m) = (1 - m)^3$
c	c	c	i	$(1 - m) \times (1 - m) \times m = m(1 - m)^2$
c	i	c	c	$(1 - m) \times m \times (1 - m) = m(1 - m)^2$
c	i	c	i	$(1 - m) \times m \times m = m^2(1 - m)$
i	c	c	c	$m \times (1 - m) \times (1 - m) = m(1 - m)^2$
i	c	c	i	$m \times (1 - m) \times m = m^2(1 - m)$
i	i	c	c	$m \times m \times (1 - m) = m^2(1 - m)$
i	i	c	i	$m \times m \times m = m^3$

The sum of these probabilities is $(1 - m)^3 + 3m(1 - m)^2 + 3m^2(1 - m) + m^3 = 1 - 3m + 3m^2 - m^3 + 3m - 6m^2 + 3m^3 + 3m^2 - 3m^3 + m^3 = 1$.

$$8.5 \quad \text{IRF}_{Bk}(\theta) = \frac{e^{a_{Bk}(\theta - b_{Bk})}}{1 + e^{a_{Bk}(\theta - b_{Bk})}} \cdot 1 = \frac{e^{a_{Bk}(\theta - b_{Bk})}}{1 + e^{a_{Bk}(\theta - b_{Bk})}} \cdot \frac{e^{-a_{Bk}(\theta - b_{Bk})}}{e^{-a_{Bk}(\theta - b_{Bk})}} = \frac{e^{a_{Bk}(\theta - b_{Bk})} e^{-a_{Bk}(\theta - b_{Bk})}}{(1 + e^{a_{Bk}(\theta - b_{Bk})}) e^{-a_{Bk}(\theta - b_{Bk})}} =$$

$$\frac{e^{a_{Bk}(\theta - b_{Bk}) - a_{Bk}(\theta - b_{Bk})}}{e^{-a_{Bk}(\theta - b_{Bk})} + e^{a_{Bk}(\theta - b_{Bk}) - a_{Bk}(\theta - b_{Bk})}} = \frac{e^0}{e^{-a_{Bk}(\theta - b_{Bk})} + e^0} = \frac{1}{e^{-a_{Bk}(\theta - b_{Bk})} + 1}.$$

Therefore,

$$\lim_{\theta \rightarrow -\infty} \text{IRF}_{Bk}(\theta) = \lim_{\theta \rightarrow -\infty} \frac{1}{e^{-a_{Bk}(\theta - b_{Bk})} + 1} = 0,$$

and

$$\lim_{\theta \rightarrow +\infty} \text{IRF}_{\text{Bk}}(\theta) = \lim_{\theta \rightarrow +\infty} \frac{1}{e^{-a_{\text{Bk}}(\theta - b_{\text{Bk}})} + 1} = 1.$$

8.6 The equation of the tangent line of Birnbaum's two-parameter item response function is the first derivative of $\text{IRF}_{\text{Bk}}(\theta)$ to θ . $\text{IRF}_{\text{Bk}}(\theta)$ is written as in Exercise 8.5:

$$\text{IRF}_{\text{Bk}}(\theta) = \frac{1}{e^{-a_{\text{Bk}}(\theta - b_{\text{Bk}})} + 1} = (1 + e^{-a_{\text{Bk}}(\theta - b_{\text{Bk}})})^{-1}.$$

The first derivative of $\text{IRF}_{\text{Bk}}(\theta)$ to θ is:

$$\frac{d}{d\theta} [\text{IRF}_{\text{Bk}}(\theta)] = \frac{d}{d\theta} [(1 + e^{-a_{\text{Bk}}(\theta - b_{\text{Bk}})})^{-1}] = -(1 + e^{-a_{\text{Bk}}(\theta - b_{\text{Bk}})})^{-2} \cdot e^{-a_{\text{Bk}}(\theta - b_{\text{Bk}})} \cdot (-a_{\text{Bk}}).$$

At $\theta = b_{\text{Bk}}$ the slope of this tangent line is:

$$-(1 + e^{-a_{\text{Bk}}(\theta - b_{\text{Bk}})})^{-2} \cdot e^{-a_{\text{Bk}}(\theta - b_{\text{Bk}})} \cdot (-a_{\text{Bk}}) = -(1 + e^0)^{-2} \cdot e^0 \cdot (-a_{\text{Bk}}) = -(1 + 1)^{-2} \cdot 1 \cdot (-a_{\text{Bk}}) = .25a_{\text{Bk}}.$$

8.7 $\text{Pr}_{j^k} = .95$, $\text{Pr}_{j^l} = .82$,

$$\text{Pr}_{j^k}(c, c) = .95 \times .82 = .78,$$

$$\text{Pr}_{j^k}(c, i) = .95 \times (1 - .82) = .17,$$

$$\text{Pr}_{j^k}(i, c) = (1 - .95) \times .82 = .04,$$

$$\text{Pr}_{j^k}(i, i) = (1 - .95) \times (1 - .82) = .01.$$

8.8 Items k and l are administered to test taker j. As in the derivation of Formula 8.9a, Test taker j's log-odds at the kth item is:

$$\text{Log - Odds}_{jk} = \theta_j - b_{\text{Rk}}$$

and, analogously, his (her) log-odds at the lth item is:

$$\text{Log - Odds}_{jl} = \theta_j - b_{\text{Rl}}$$

Therefore, the difference of the log-odds of the two items is:

$$\text{Log - Odds}_{jk} - \text{Log - Odds}_{jl} = (\theta_j - b_{Rk}) - (\theta_j - b_{Rl}) = \theta_j - b_{Rk} - \theta_j + b_{Rl} = b_{Rl} - b_{Rk}. \quad (8.9b)$$

8.9 It follows from Formula 8.7a that test taker j's probability of giving the correct (yes) answer to the kth item is:

$$\Pr_{jk}(c) = \text{IRF}_{Bk}(\Theta = \theta_j) = \frac{e^{a_{Bk}(\theta_j - b_{Bk})}}{1 + e^{a_{Bk}(\theta_j - b_{Bk})}},$$

and his (her) probability of giving the incorrect (no) answer is:

$$\Pr_{jk}(i) = 1 - \Pr_{jk}(c) = \frac{1 + e^{a_{Bk}(\theta_j - b_{Bk})}}{1 + e^{a_{Bk}(\theta_j - b_{Bk})}} - \frac{e^{a_{Bk}(\theta_j - b_{Bk})}}{1 + e^{a_{Bk}(\theta_j - b_{Bk})}} = \frac{1}{1 + e^{a_{Bk}(\theta_j - b_{Bk})}}.$$

Therefore, Test taker j's odds is:

$$\text{Odds}_{jk} = \frac{\Pr_{jk}(c)}{\Pr_{jk}(i)} = \Pr_{jk}(c) \cdot \left\{ \Pr_{jk}(i) \right\}^{-1} = \frac{e^{a_{Bk}(\theta_j - b_{Bk})}}{1 + e^{a_{Bk}(\theta_j - b_{Bk})}} \cdot (1 + e^{a_{Bk}(\theta_j - b_{Bk})}) = e^{a_{Bk}(\theta_j - b_{Bk})},$$

and his (her) log-odds at the kth item is:

$\text{Log - Odds}_{jk} = \ln e^{a_{Bk}(\theta_j - b_{Bk})} = a_{Bk}(\theta_j - b_{Bk})$. In the same way Test taker j's log-odds at the kth item is:

$\text{Log - Odds}_{jk} = a_{Bk}(\theta_j - b_{Bk})$. Therefore, the difference of the log-odds of the two test takers at the kth item is:

$$\text{Log - Odds}_{jk} - \text{Log - Odds}_{j'k} = a_{Bk}(\theta_j - b_{Bk}) - a_{Bk}(\theta_{j'} - b_{Bk}) = a_{Bk}(\theta_j - \theta_{j'}).$$

The comparison of Test takers j and j' involves the item parameter a_{Bk} , which means that Birnbaum's two-parameter logistic model lacks the property of specific objectivity.

8.10 $\Pr_1(\mathbf{c}) = \Pr_2(\mathbf{c}) = \frac{2}{10} = .2 < \Pr_3(\mathbf{c}) = \Pr_4(\mathbf{c}) = \frac{8}{10} = .8$, and $\Pr_{12}(\mathbf{i}, \mathbf{c}) = \Pr_{34}(\mathbf{i}, \mathbf{c}) = \frac{0}{10} = 0$.

It follows from the first part of Formula 8.11 ($\Pr_1(\mathbf{c}) = \Pr_2(\mathbf{c})$) that

$$H_{12} = 1 - \frac{0}{(1 - .2) \times .2} = 1 - 0 = 1,$$

and in the same way that $H_{34} = 1$.

$$\Pr_{13}(\mathbf{c}, \mathbf{i}) = \Pr_{14}(\mathbf{c}, \mathbf{i}) = \Pr_{23}(\mathbf{c}, \mathbf{i}) = \Pr_{24}(\mathbf{c}, \mathbf{i}) = \frac{0}{10} = 0.$$

It follows from the second part of Formula 8.11 ($\Pr_1(\mathbf{c}) < \Pr_3(\mathbf{c})$) that

$$H_{13} = 1 - \frac{0}{.2 \times (1 - .8)} = 1 - 0 = 1,$$

and in the same way that

$$H_{14} = H_{23} = H_{24} = 1.$$

Chapter 9

9.1

Five-category item	Splitting			
	(a)	(b)	(c)	(d)
$Pr_{jk}(\text{cat.1}) = .1$.1	.1	.1	.1
$Pr_{jk}(\text{cat.2}) = .2$.2	+	+	+
$Pr_{jk}(\text{cat.3}) = .3$	+	.2 = .3	.2	.2
$Pr_{jk}(\text{cat.4}) = .3$.3	+	+	+
$Pr_{jk}(\text{cat.5}) = .1$	+	.3	.3 = .6	.3
	.3	+	+	+
	+	.3	.3	.3 = .9
	.1 = .9	+	+	.1
		.1 = .7	.1 = .4	

9.2 The two IRFs intersect at the point where $IRF_{Bk(a)} = IRF_{Bk(b)}$. These two IRFs are equal if

$$\begin{aligned}
 a_{Bk(a)}(\theta - b_{Bk(a)}) &= a_{Bk(b)}(\theta - b_{Bk(b)}) \Rightarrow \\
 a_{Bk(a)}\theta - a_{Bk(a)}b_{Bk(a)} &= a_{Bk(b)}\theta - a_{Bk(b)}b_{Bk(b)} \Rightarrow \\
 a_{Bk(a)}\theta - a_{Bk(b)}\theta &= a_{Bk(a)}b_{Bk(a)} - a_{Bk(b)}b_{Bk(b)} \Rightarrow \\
 (a_{Bk(a)} - a_{Bk(b)})\theta &= a_{Bk(a)}b_{Bk(a)} - a_{Bk(b)}b_{Bk(b)}.
 \end{aligned}$$

Therefore, the IRFs intersect at the point

$$\theta_{\text{int}} = \frac{a_{Bk(a)}b_{Bk(a)} - a_{Bk(b)}b_{Bk(b)}}{a_{Bk(a)} - a_{Bk(b)}}.$$

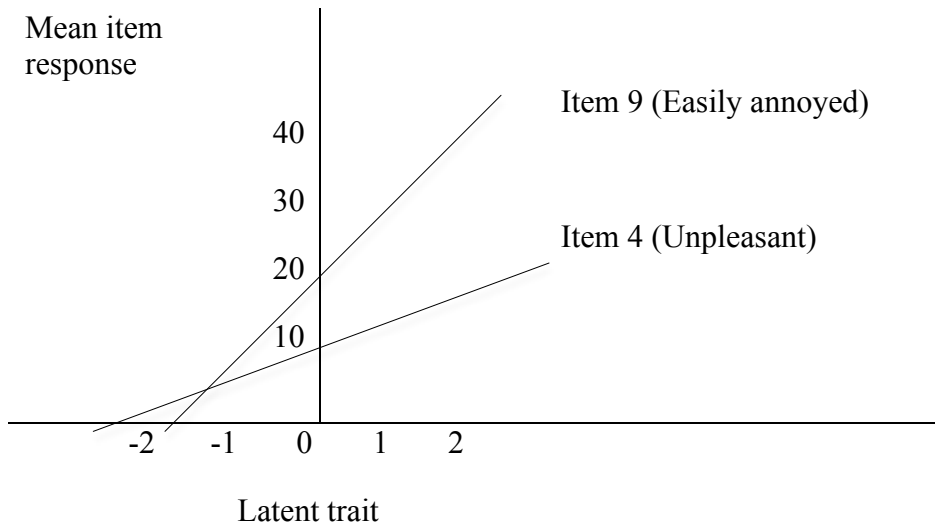
9.4 The equation of the tangent line of Birnbaum's three-parameter item response function is the derivative of $IRF_{BTK}(\theta)$ to θ :

$$\begin{aligned}
 \frac{d}{d\theta} [IRF_{BTK}(\theta)] &= (\text{see Exercise 8.6}) \\
 \frac{d}{d\theta} [g_k + (1 - g_k)(1 + e^{-a_{BTK}(\theta - b_{BTK})})^{-1}] &= (1 - g_k) \frac{d}{d\theta} [(1 + e^{-a_{BTK}(\theta - b_{BTK})})^{-1}].
 \end{aligned}$$

Analogously to Exercise 8.6, it is shown that the slope of this tangent line at $\theta = b_{BTK}$ is :

$$(1 - g_k) \times .25a_{BTK}.$$

9.5



Chapter 10

10.1 (a) Information of the 7-item test is:

$$.44 + .22 + .54 + .58 + .83 + .57 + .89 = 4.07;$$

(b) The latent trait reliability of the 7-item test is:

$$\frac{1 \times 4.07}{1 + 1 \times 4.07} = .80.$$

10.2 At $\Theta = 0$ the test information is 4.3 (Table 10.2), and the estimated standard error is:

$$\sqrt{\frac{1}{4.3}} = .48.$$

The estimated 90% confidence interval of an estimated latent trait value of 0 is from 0 -

$$1.65 \times .48 = -.79 \text{ to } 0 + 1.65 \times .48 = .79.$$

10.3 (a) The DIF is unidirectional if

$$a_{BM3} = a_{BF3} \text{ and } b_{BM3} \neq b_{BF3}.$$

(b) The DIF is bidirectional if

$$a_{BM3} \neq a_{BF3}.$$

Chapter 12

12.1 The sample mean of the Z-scores is:

$$\bar{Z} = \frac{1}{N_s} \sum_{j=1}^{N_s} Z_j = \frac{1}{N_s} \sum_{j=1}^{N_s} \frac{TS_j - \overline{TS}}{s_{TS}} = \frac{1}{N_s s_{TS}} \sum_{j=1}^{N_s} (TS_j - \overline{TS}) = \frac{1}{N_s s_{TS}} \left(\sum_{j=1}^{N_s} TS_j - \sum_{j=1}^{N_s} \overline{TS} \right) = \text{(because}$$

$$\sum_{j=1}^{N_s} TS_j = N_s \overline{TS} \text{ and } \sum_{j=1}^{N_s} \overline{TS} = N_s \overline{TS})$$

$$\frac{1}{N_s s_{TS}} (N_s \overline{TS} - N_s \overline{TS}) = 0.$$

The sample variance of the Z-scores is:

$$s_Z^2 = \frac{1}{N_s} \sum_{j=1}^{N_s} (Z_j - \bar{Z})^2 = \text{(because } \bar{Z} = 0)$$

$$\frac{1}{N_s} \sum_{j=1}^{N_s} Z_j^2 = \frac{1}{N_s} \sum_{j=1}^{N_s} \left\{ \frac{TS_j - \overline{TS}}{s_{TS}} \right\}^2 = \frac{1}{s_{TS}^2} \times \frac{1}{N_s} \sum_{j=1}^{N_s} (TS_j - \overline{TS})^2 = \frac{1}{s_{TS}^2} s_{TS}^2 = 1.$$

12.2 The sample mean of the T-scores is:

$$\bar{T} = \frac{1}{N_s} \sum_{j=1}^{N_s} T_j = \frac{1}{N_s} \sum_{j=1}^{N_s} (10Z_j + 50) = \frac{1}{N_s} \left(\sum_{j=1}^{N_s} 10Z_j + \sum_{j=1}^{N_s} 50 \right) = \frac{1}{N_s} \left(10 \sum_{j=1}^{N_s} Z_j + \sum_{j=1}^{N_s} 50 \right) =$$

$$\text{(because } \sum_{j=1}^{N_s} Z_j = N_s \bar{Z} \text{ and } \sum_{j=1}^{N_s} 50 = 50N_s) \frac{1}{N_s} (10 \times N_s \bar{Z} + 50N_s) =$$

$$\text{(because } \bar{Z} = 0) \frac{1}{N_s} \times 50N_s = 50.$$

The sample variance of the T-scores is:

$$s_T^2 = \frac{1}{N_s} \sum_{j=1}^{N_s} (T_j - \bar{T})^2 = \text{(because } \bar{T} = 50)$$

$$\frac{1}{N_s} \sum_{j=1}^{N_s} (T_j - 50)^2 = \frac{1}{N_s} \sum_{j=1}^{N_s} (10Z_j + 50 - 50)^2 = \frac{1}{N_s} \sum_{j=1}^{N_s} (10Z_j)^2 = 100 \times \frac{1}{N_s} \sum_{j=1}^{N_s} Z_j^2 =$$

$$100 \times s_Z^2 = 100 \times 1 = 100.$$

Therefore, the standard deviation of the T-scores is

$$s_T = \sqrt{100} = 10.$$

12.3 The z-score of this girl is

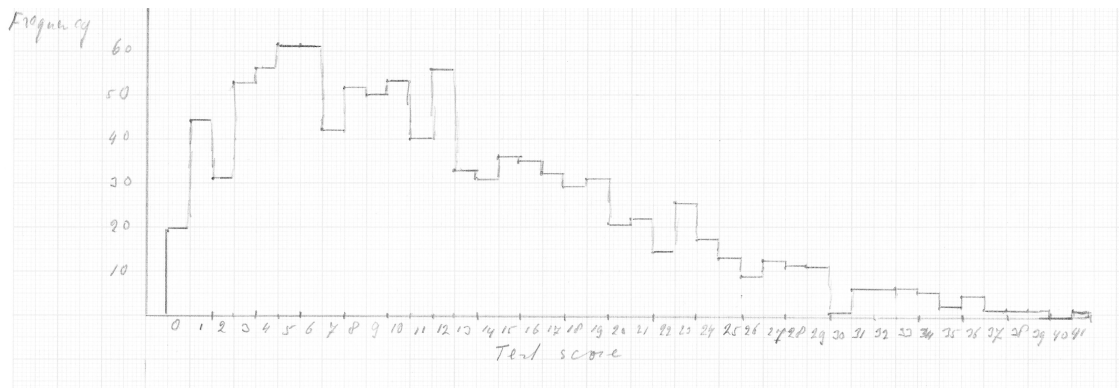
$$Z = \frac{11 - 8}{8.23} = .36,$$

and her T-score is

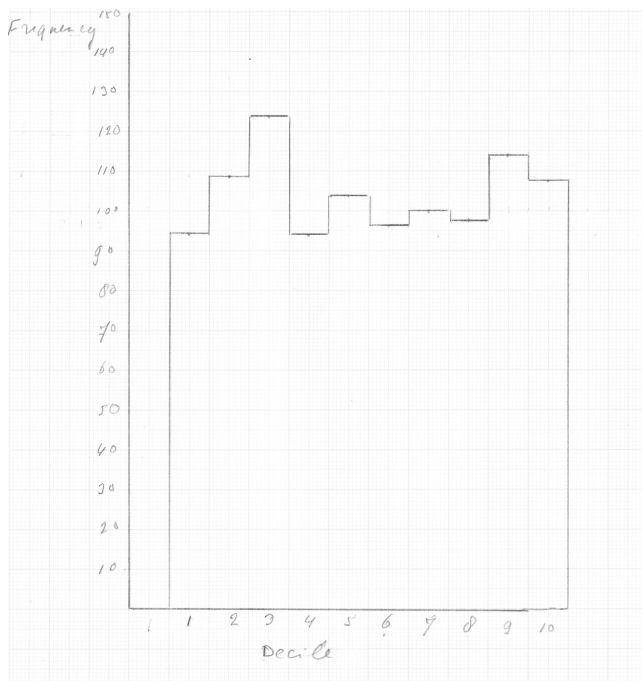
$$T = 10 \times .36 + 50 = 53.6.$$

It follows from Table 12.1 that her percentile score is 52, her decile score is 6, and her stanine score is 5.

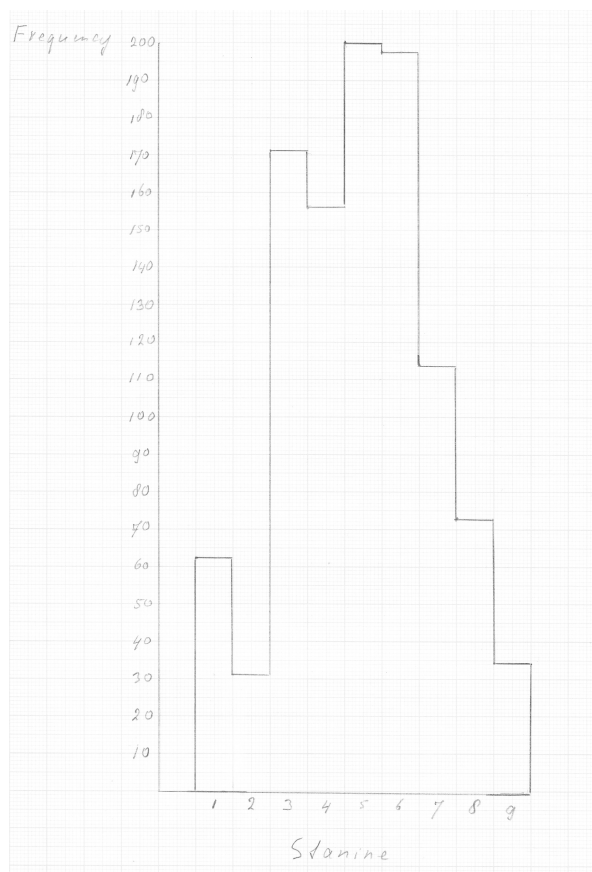
12.4 (a) Observed test scores



(b) Decile scores



(c) Stanine scores



12.5 The girl's difference score is

$$D_j = 3 - 11 = -8.$$

The estimated standard error of measurement of the test is $\hat{S}_{em} = 2.6$. Therefore, an approximation of the standard error of the girl's difference score is

$$\hat{S}_{em_j}(D) \approx \hat{S}_{em}\sqrt{2} = 2.6\sqrt{2} = 3.68.$$

The estimated 90% confidence interval of the girl's difference score is from $-8 - 1.65 \times 3.68 = -14.07$ to $-8 + 1.65 \times 3.68 = -1.93$. The null hypothesis of no change from pretest to posttest is rejected at the 10% significance level because the 90% confidence interval does not contain the value 0.

12.6 $\text{Var}_r(\hat{\delta}) = \text{Var}_r(\hat{\theta}_{2j} - \hat{\theta}_{1j}) = \text{Var}_r(\hat{\theta}_{2j}) - 2\text{Cov}_r(\hat{\theta}_{2j}, \hat{\theta}_{1j}) + \text{Var}_r(\hat{\theta}_{1j}) =$ (because of the assumption of independence of $\hat{\theta}_{1j}$ and $\hat{\theta}_{2j}$)

$$\text{Var}_r(\hat{\theta}_{2j}) - 2 \times 0 + \text{Var}_r(\hat{\theta}_{1j}) = \text{Var}_r(\hat{\theta}_{1j}) + \text{Var}_r(\hat{\theta}_{2j}).$$

12.7 The test taker's estimated latent difference score is $-1.0 - 1.0 = -2.0$. Under Birnbaum's two-parameter model the estimated standard error of his latent difference score is (see Table 10.2)

$$\sqrt{\frac{1}{4.7} + \frac{1}{2.4}} = .79.$$

The estimated 90% confidence interval of the test taker's latent difference score is from $-2.0 - 1.65 \times .79 = -3.30$ to $-2.0 + 1.65 \times .79 = -.70$. The null hypothesis of no change from pretest to posttest is rejected at the 10% significance level because the confidence interval does not contain the value 0.

12.8 At $\theta_{\text{cut}} = .19$ the student's probability of giving the correct answer is

$$\text{Pr}(c) = \frac{e^{\theta_{\text{cut}} - b_R}}{1 + e^{\theta_{\text{cut}} - b_R}} = \frac{e^{.19 - b_R}}{1 + e^{.19 - b_R}}.$$

Applying the log-odds transformation yields

$$.19 - b_R = \ln \frac{1/2}{1/2} = \ln 1 = 0 \Rightarrow b_R = .19.$$

Chapter 13

$$13.1 \quad \text{Cr}' = \frac{e^{\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p}}{1 + e^{\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p}} \times 1 = \frac{e^{\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p}}{1 + e^{\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p}} \times \frac{e^{-(\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p)}}{e^{-(\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p)}} =$$

$$\frac{e^{\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p} e^{-(\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p)}}{(1 + e^{\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p}) e^{-(\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p)}} =$$

$$\frac{e^{\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p} e^{-(\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p)}}{e^{-(\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p)} + e^{\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p} e^{-(\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p)}} =$$

$$\frac{e^0}{e^{-(\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p)} + e^0} = \frac{1}{e^{-(\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \dots + \beta_p P_p)} + 1}$$

If in the limit the part of the denominator between parentheses goes to plus infinity $Cr' = 1$, and if this part goes to minus infinity $Cr' = 0$.

13.2 The student's predicted psychophysiology performance is

$$\frac{e^{-4.71 + .11 \times 30 + .08 \times 25}}{1 + e^{-4.71 + .11 \times 30 + .08 \times 25}} = .64.$$

The student's predicted performance is larger than .5, and it is predicted that the student will pass the psychophysiology examination.

13.3 The students' predicted college graduation/drop out are:

$$\text{Student 1: } \frac{e^{-3.5 + 1 \times 2.5}}{1 + e^{-3.5 + 1 \times 2.5}} = .27 \text{ (smaller than .5, i.e., drop out is predicted);}$$

$$\text{students 2 and 3: } \frac{e^{-3.5 + 1 \times 3}}{1 + e^{-3.5 + 1 \times 3}} = .38 \text{ (smaller than .5, i.e., drop out is predicted);}$$

$$\text{students 4, 5, 6, and 7: } \frac{e^{-3.5 + 1 \times 4}}{1 + e^{-3.5 + 1 \times 4}} = .62 \text{ (larger than .5, i.e., graduation is predicted);}$$

$$\text{students 8, 9, and 10: } \frac{e^{-3.5 + 1 \times 4.5}}{1 + e^{-3.5 + 1 \times 4.5}} = .73 \text{ (larger than .5, i.e., graduation is predicted).}$$

The number of students that graduate (6) is larger than the number of students (4) that drop out. Therefore, the best guess is that all ten students will graduate, which means that the criterion-based number of misclassifications is 4 (students 1, 3, 4, and 5 are predicted to

graduate, but they drop out). Using the GPA and the logistic regression function 3 misclassifications are made: student 2 is predicted to drop out, but graduated, and students 4 and 5 are predicted to graduate, but dropped out. Therefore,

$$PAD = \frac{4 - 3}{4} = .25.$$

13.4 The regression lines Formulas (13.9a) and (13.9b) intersect at the point P_{1int} . At this intersection point the equations of the lines are equal, that is.

$$\beta_{0A} + \beta_{1A}P_{1int} = \beta_{0B} + \beta_{1B}P_{1int} \Rightarrow \beta_{1A}P_{1int} - \beta_{1B}P_{1int} = \beta_{0B} - \beta_{0A} \Rightarrow$$

$$P_{1int}(\beta_{1A} - \beta_{1B}) = \beta_{0B} - \beta_{0A} \Rightarrow P_{1int} = \frac{\beta_{0B} - \beta_{0A}}{\beta_{1A} - \beta_{1B}}.$$

13.5 (a) Models 1 and 3:

$$H_0(int): \beta_{0A} = \beta_{0B} = \beta.$$

(b) Models 2 and 3:

$$H_0(res): \sigma_A^2 = \sigma_B^2 = \sigma^2.$$

(c) Models 4 and 5:

$$H_0(res): \sigma_A^2 = \sigma_B^2 = \sigma^2.$$

$$13.6 \quad Cr' = \frac{e^{\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \beta_3 P_3 + \beta_4 P_1 P_3 + \beta_5 P_2 P_3 + \beta_6 P_1 P_2 P_3}}{1 + e^{\beta_0 + \beta_1 P_1 + \beta_2 P_2 + \beta_3 P_3 + \beta_4 P_1 P_3 + \beta_5 P_2 P_3 + \beta_6 P_1 P_2 P_3}},$$

where P_1 and P_2 are the two predictors, and P_3 is the dummy coded group membership variable (e.g., Group A is coded 1 and Group B is coded 0).

$$13.7 \quad LOS_1 = \hat{\beta}_0 - .03 \times 1 + .09 \times 4 + .37 \times 3 + .22 \times 4 + .09 \times 0 = \hat{\beta}_0 + 2.32;$$

$$LOS_2 = \hat{\beta}_0 - .03 \times 0 + .09 \times 4 + .37 \times 4 + .22 \times 3 + .09 \times 1 = \hat{\beta}_0 + 2.59.$$

The second patient is selected because (s)he has the largest predicted stay.

13.8 (a) The proportions of misclassifications are per selection ratio:

- Selection ratio .51: $\hat{P}_{FN} + \hat{P}_{FP} = .06 + .21 = .27$;

- selection ratio .79: $\hat{P}_{FN} + \hat{P}_{FP} = .02 + .44 = .46$;

- selection ratio 1: $\hat{P}_{FN} + \hat{P}_{FP} = 0 + .63 = .63$.

Selection ratio .51 is the best of the three selection ratios because it has the smallest proportion of misclassifications.

(b) The weighted sum of the proportions correct classifications per selection ratio is:

- Selection ratio .51: $\hat{P}_{TP} + 10\hat{P}_{TN} = .30 + 10 \times .43 = 4.60$;

- selection ratio .79: $\hat{P}_{TP} + 10\hat{P}_{TN} = .35 + 10 \times .19 = 2.25$;

- selection ratio 1: $\hat{P}_{TP} + 10\hat{P}_{TN} = .37 + 10 \times 0 = .37$.

Selection ratio .51 is the best of the three selection ratios because it yields the largest weighted sum.

13.9

Creoles		Asians	
selection ratio	$2\hat{P}_{CTP} + \hat{P}_{CTN}$	selection ratio	$\hat{P}_{ATP} + \hat{P}_{ATN}$
169/169	1.4792	124/124	.6855
168/169	1.4851	123/124	.6936
167/169	1.4910*	122/124	.6855
163/169	1.4793	121/124	.6935
159/169	1.4852	120/124	.7016*
155/169	1.4571	119/124	.6936
150/169	1.4320	117/124	.6774
144/169	1.3964	114/124	.6855

The optimal quota-free selection ratio is for Creoles 167/169 and for Asians 120/124.

13.10

Frequency			$\frac{169}{293} \times 2(\hat{P}_{CTP} + \hat{P}_{CTN}) + \frac{124}{293} \times (\hat{P}_{ATP} + \hat{P}_{ATN})$
	Creoles	Asian	
	4	0	.4608*
	3	1	.4575
	2	2	.4539
	1	3	.4505
	0	4	.4472

All four grants have to be awarded to the Creole children.

Chapter 14

14.1 Item no.	2	1	3	5	4
Test taker j's response pattern	c	c	i	i	c

The number of Guttman errors is 2 (incorrect answer to Item 3 precedes correct answer to Item 4, and incorrect answer to Item 5 precedes correct answer to Item 4):

$$PFG_j = 2.$$

14.2

Item no.	Test taker no.		
	1	2	3
1	a	a	a
2	a	a	a
3	d	d	a
4	a	d	d
5	d	a	d
6	a	d	a
7	a	a	d
8	d	d	a
9	a	d	d
10	a	a	d
$\Pr^T(a)$.7	.5	.5

The proportion of items where the first test taker gives a d-answer and the third test taker gives an a-answer is $\frac{2}{10} = .2$; the proportion of items where the second test taker gives a d-answer and the third test taker gives an a-answer is $\frac{3}{10} = .3$. Therefore,

$$\text{PFM}_3 = H_3^T = 1 - \frac{.2 + .3}{(1 - .7) \times .5 + .5(1 - .5)} = 1 - \frac{.5}{.4} = -.25.$$

14.3 The first 3 items are easier than the last 3 items. Therefore, it is most likely that a test taker who answers correctly 3 items will answer correctly the first 3 easiest items (pattern no. 1), and it is least likely that he or she will answer correctly the last 3 most difficult items (pattern no. 20).

14.4 Item response pattern

- (1) (c, i, i, i, i, i)
- (2) (i, c, i, i, i, i)
- (3) (i, i, c, i, i, i)
- (4) (i, i, i, c, i, i)
- (5) (i, i, i, i, c, i)

(6) (i, i, i, i, i, c)

The PRF-values per pattern are:

(1) $1 \times (-1.5) + 0 \times (-1) + 0 \times (-.5) + 0 \times (.5) + 0 \times 1 + 0 \times 1.5 = -1.5,$

(2) $0 \times (-1.5) + 1 \times (-1) + 0 \times (-.5) + 0 \times (.5) + 0 \times 1 + 0 \times 1.5 = -1,$

(3) $-.5,$

(4) $.5,$

(5) $1,$

(6) $1.5.$

14.5 At each step of CAT an item is selected that yields maximum information on the test taker's latent trait value that is estimated in the previous step of CAT. Under Jöreskog's congeneric model the item information is a constant that does not vary with the latent trait (Formula 10.12). Therefore, an item gives the same information at all latent trait values, which implies that it makes no sense to search for an item that gives maximum information at a given latent trait estimate.

14.6 The 95% confidence interval of j's latent trait value is from $.6 - 1.96 \times .5 = -.38$ to $.6 + 1.96 \times .5 = 1.58$. It is decided that the student passes the test because the cutting score $\theta_{\text{cut}} = -.5$ is below the lower bound of the confidence interval $(-.38, 1.58)$.